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Betweenness of membership functions: classical case and hyperbolic-valued functions

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Abstract. We study betweenness of membership functions in the fuzzy setting and for membership functions taking values in the set of hyperbolic numbers.

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1. Introduction

Prologue: To develop mathematical tools to study similarity of objects or situations is a very important problem in a wide range of topics, from botany to psychology and more, and involve in particular comparisons of sets, finite or infinite. We mention for instance Paul Jaccard [1], whose studies of comparative floral distribution lead to the notion of Jaccard index of similarity (*coefficient de communauté*, in French). To define this index, some notations need to be introduced. Given a set Ω, we denote by $\Omega \setminus A$ the *complement* of $A \in \mathcal{P}(\Omega)$ in Ω and by $A\Delta B$ the *symmetric difference* of A and B:

$$A\Delta B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B). \tag{1.1}$$

The *Jaccard index* is defined by

$$J(A,B) = \frac{\text{Card } A \cap B}{\text{Card } A \cup B}, \tag{1.2}$$

where A and B are two finite subsets (not both empty) of a common set Ω. If $A = B = \varnothing$, one sets J to be 0. Note that $J(A,B) \in [0,1]$ and that

$$D(A,B) = 1 - J(A,B) = \frac{\text{Card } A \cup B - \text{Card } A \cap B}{\text{Card } A \cup B} = \frac{\text{Card } A\Delta B}{\text{Card } A \cup B} \tag{1.3}$$

is a well known distance on sets, (See Definition 2.1 for the notion of distance) similar to the distance between finite random variables introduced earlier in information theory by C. Rajski; see e.g. [2] and [3] for the latter. The proofs in these papers are easily adapted to the case of finite sets. We also note the works [4] and [5] (in Polish; available online) of Marczewski and Steinhaus, where (1.3) (and counterparts for functions) is investigated, with applications to study of species growing in forests (analysis of biotopes). Note that these authors do not mention Jaccard. For another later proof of the triangle inequality for the Jaccard distance, see e.g. [6].

We also mention the work of Amos Tversky [7], where a representation theorem to measure similarity between different sets of objects is developed using the decomposition of $A \cup B$ into the non-overlapping sets $A \setminus B$, $B \setminus A$ and $A \Delta B$. For a recent survey we refer to [8].

Betweenness: In the study of similarity of sets, estimating the betweenness of a set of features with respect to two other sets of features is a major question, which can be defined and studied in different ways, depending on the underlying structure. It involves important analytic tools, such as metric spaces, strictly convex norms and lattices. In a general metric space one can define the notion as corresponding to cases of equality in the triangle inequality. In a vector space it is easy to define betweenness: a vector is between two vectors u and v if it belongs to the closed interval defined by these two vectors (or equal to u when $u = v$). In [9], Restle defines and studies the notion of betweenness of sets. Let Ω be a set and let $A, B, C \in \mathcal{P}(\Omega)$. Following Restle (See [9, Definition 2 p. 210]) one says that the set C is between the sets A and B if (Restle writes these two inclusion conditions in a slightly different, but equivalent, way)

$$A \cap B \subset C \subset A \cup B, \tag{1.4}$$

which can be translated in terms of indicator functions (See (4.1)) as

$$1_{A\cap B}(x) \leq 1_C(x) \leq 1_{A\cup B}(x). \tag{1.5}$$

Among other questions Restle is interested in [9] in the case of equality in the triangle inequality in an underlying metric space. To palliate the lack of vector space structure Restle introduces the notion of linear array of sets.

The paper: In the first part of the present paper we study the counterpart of some aspects of Restle's paper in the fuzzy sets theory setting, when indicator functions of sets are replaced by membership functions, whose definition we now recall (See for instance [10, 11]):

Definition 1.1. A function from X into $[0, 1]$, i.e. belonging to $[0, 1]^X$, is called a *membership function.*

We write a membership function f as $\mu_{\widetilde{A}}$, where by definition, $\widetilde{A}$ denotes the fuzzy set defined by f.

In machine learning, a recent research trend consists in replacing the real numbers by hypercomplex numbers; see for instance [12, 13] for complex numbers, [14] for bicomplex numbers and [15] for hyperbolic numbers. In the second part of this paper, and inspired by the work [16] where probabilities are allowed to take values in the set of hyperbolic numbers (we will say for short, hyperbolic-valued), we initiate a study of fuzzy set theory when the membership function is hyperbolic-valued. Definitions are recalled in the sequel, but we already mention at this stage that hyperbolic numbers can be seen as the set of matrices of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, where a and b run through the real numbers.

We therefore address two different audiences, the fuzzy set community and the hypercomplex analysis community, and will review materials from both fields to make the paper accessible to both groups.

To pursue we recall that for two (not necessarily Hermitian) matrices A and B in $\mathbb{C}^{n\times n}$ one says that $A \le B$ if $B - A$ is a positive semi-definite matrix (one also says non-negative), i.e. if $B - A$ is Hermitian (symmetric in the case of matrices with real entries) with non-negative eigenvalues.

Two Hermitian matrices which do not commute cannot be simultaneously diagonalized and one cannot define in a natural way their maximum and minimum using the natural order of matrices. On the other hand, hyperbolic numbers are simultaneously diagonalizable and they form a lattice: we can define maximum and minimum (with respect to the above partial order) of any pair of hyperbolic numbers in the set of hyperbolic numbers. As a consequence we can extend to the hyperbolic setting important operations on fuzzy sets which involve maximum and minimum.

As mentioned above in fuzzy set theory one replaces indicator functions of subsets of a given set X by functions from X into $[0, 1]$. We introduce a new operator on membership functions: given f and g two membership functions we associate the hyperbolic-valued function

$$M_{f,g}(x) = \frac{1}{2}\begin{pmatrix} f(x) + g(x) & f(x) - g(x) \\ f(x) - g(x) & f(x) + g(x) \end{pmatrix}, \quad x \in X. \tag{1.6}$$

Formula (1.6) defines a new operation on membership functions, and $M_{f,g}$ takes values in the counterpart of $[0, 1]$ for hyperbolic numbers. The main properties of this operation are obtained using the fact that the hyperbolic numbers form a lattice.

We note (See Section 5 for definitions) that already in classical fuzzy set theory, fuzzy sets have been generalized to sets defined by two membership functions (intuitionistic fuzzy sets, also known as bipolar fuzzy sets, and soft fuzzy sets). The present extension is different from these approaches.

The hyperbolic numbers form a commuting family of Hermitian matrices, and as such is simultaneously diagonalizable, as is also immediately seen from (8.1). More generally recall that a commuting family of complex matrices is simultaneously triangularizable; see [17]. The present theory could be extended to families of commuting symmetric matrices, or diagonalizable familes of non-symmetric matrices.

The paper consists of ten sections besides the introduction. In Section 2 we discuss distances associated to positive definite kernels. In Section 3 we discuss betweenness of vectors in a vector space. Betweenness of sets is studied in Section 4. A few facts from fuzzy set theory are reviewed in Section 5. Betweenness in the fuzzy setting is studied in Sections 6 and 7, using two different approaches: characterization in terms of intervals and in terms of strong α-cuts. That the two defintions are equivalent is proved in Theorem 7.2. The definition and main properties of hyperbolic numbers are reviewed in Section 8, while hyperbolic-valued membership functions are studied in Section 9 and their properties in Section 10. Betweenness in the setting of hyperbolic-valued membership functions is considered in Section 11.

Finally, a word on notation: $a \wedge b$ and $a \vee b$ denote respectively the minimum and maximum of the real numbers a and b, and more genrally the corresponding operations in a lattice. The matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

will be denoted sometimes by 0 for the first and by 1 or I_2 for the second.

2. Positive definite kernel and associated metric

We review some facts on positive definite functions relevant to the present work; for further references we suggest [18, 19, 20]. For completeness we recall:

Definition 2.1. Let E be a set. The map d from $E \times E$ into $[0, \infty)$ is called a *metric* (or a *distance*), if the following three conditions hold for all $x, y, z \in E$,

$$d(x,y) = 0 \iff x = y \tag{2.1}$$

$$d(x,y) = d(y,x) \tag{2.2}$$

$$d(x,y) \leq d(x,z) + d(y,z). \tag{2.3}$$

(2.3) is called the *triangle inequality* and the pair (E, d) (or E for short) is called a *metric space*.

Positive definite kernels (we will also say positive definite functions, although the latter terminology is usually used for a smaller class of kernels) whose definition we now recall, play an important role in machine learning, in particular in the theory of support vector machines; see [21] for a recent account. Here they are of special interest because of the metric induced on the set where such a function is defined; see [22] and see [18, 19, 20] for more information on positive definite kernels.

Definition 2.2. Let E be a set and let $K(t,s)$ be defined on $E \times E$. It is called *positive definite* on E, if for every choice of $N \in \mathbb{N}$ and $t_1, \ldots, t_N \in E$ the matrix $(K(t_\ell, t_j))_{\ell,j=1}^N$ is positive semi-definite.

The following classical theorem gives a characterization of positive definite functions; one direction is quite clear and in the other one can take $\mathcal{H}$ to be the reproducing kernel Hilbert space $\mathcal{H}(K)$ with reproducing kernel $K(t,s)$ since

$$K(t,s) = \langle K(\cdot, s), K(\cdot, t) \rangle_{\mathcal{H}(K)}.$$

Theorem 2.3. *The function $K(t,s)$ is positive definite on E if and only if it factors via a Hilbert space, i.e. if and only if there exists a Hilbert space $\mathcal{H}$ and a function f_t from E into $\mathcal{H}$ such that*

$$K(t,s) = \langle f_s, f_t \rangle_{\mathcal{H}}, \quad t, s \in E. \tag{2.4}$$

For the following proposition see for instance [22], where some explicit examples are also computed.

Proposition 2.4. *Let $K(t,s)$ be positive definite on E with factorization (2.4), and assume that*

$$t \neq s \quad \Longleftrightarrow \quad f_t \neq f_s. \tag{2.5}$$

Then

$$d_K(t,s) = \sqrt{K(t,t) + K(t,s) - 2\mathrm{Re}\, K(t,s)} \tag{2.6}$$

defines a distance on E.

Proof. In view of (2.4) we have

$$d_K(t,s) = \|f_t - f_s\|_{\mathcal{H}} \tag{2.7}$$

and the three conditions for a metric follow, the first one using (2.5). □

Remark 2.5. For $f_t = K(\cdot,t)$ the condition (2.5) becomes

$$t \neq s \quad \Longleftrightarrow \quad K(\cdot,s) \not\equiv K(\cdot,t).$$

which is in fact a necessary and sufficient condition for (8.20) to define a metric.

As an example of metric d_K, let Ω be a finite set and let Card denote the counting measure. Then, the function

$$K(A,B) = \mathrm{Card}\,(A \cap B)$$

is positive definite on $\mathcal{P}(\Omega)$, and the associated metric is given by

$$d(A,B) = \sqrt{\mathrm{Card}\, A + \mathrm{Card}\, B - \mathrm{Card}\,(A\cap B)} = \sqrt{\mathrm{Card}\, A\Delta B}, \quad A,B \in \mathcal{P}(\Omega). \tag{2.8}$$

In general the square of a metric is not a metric, but the squareroot of a metric is still a metric. In the present case, it so happens that the square of $d(A,B)$, namely

$$d^2(A,B) = \mathrm{Card}\, A + \mathrm{Card}\, B - \mathrm{Card}\,(A\cap B) = \mathrm{Card}\, A\Delta B, \quad A,B \in \mathcal{P}(\Omega), \tag{2.9}$$

is still a metric (not induced by a positive definite kernel); see Proposition 4.3. A weighted form of d^2 appear already in [23, p. 290-291] in the study of the difference (called in [23] implicational difference) between traits in an individual. The distance $d^2(A,B)$ play a key role in the present work, and an important difference between d and d^2 will be shown in the paper.

3. Betweenness of vectors

Let $\mathcal{V}$ be a real or complex vector space, and let $u,v \in \mathcal{V}$. Recall that the interval defined by u and v is the set of vectors of the form

$$c(t) = u + t(v-u), \quad t \in [0,1] \tag{3.1}$$

which reduces to one point when $u = v$ (no order is assumed, and we can speak for instance of the interval $[0,1]$ as well as the interval $[1,0]$). It is natural to define:

Definition 3.1. The vector w is said to *be between u and v*, if $w \in [u,v]$.

Remark 3.2. When extra structure is given on $\mathcal{V}$, or for algebraic structures different from a vector space structure, the above definition need not be possible, or even if possible, need not be the best one. For instance, in case of a lattice, a natural definition would be to replace (3.1) by

$$u \wedge v + t(u \vee v - u \wedge v), \quad t \in [0,1].$$

When moreover a commutative product is available (as in the case of the hyperbolic numbers), one can replace $[0,1]$ by its counterpart with respect to the partial order; see Definition 8.5.

Recall that a norm $\|\cdot\|$ on a vector space defines a metric via

$$d(u,v) = \|u - v\|.$$

Proposition 3.3. *Let $(\mathcal{V}, \|\cdot\|)$ be a normed space and let w be between u and v. Then equality holds in the triangle inequality for d, i.e.*

$$d(u,v) = d(u,w) + d(w,v). \tag{3.2}$$

Proof. We write $w = c(t)$, where $t \in [0,1]$. Then we have

$$\begin{aligned} d(u,v) &= \|u - v\|, \\ d(u,w) &= \|u - (u - t(v-u))\| \\ &= t\|u - v\|, \\ d(w,v) &= \|u + t(v-u) - v\| \\ &= \|(1-t)(u-v)\| \\ &= (1-t)\|u - v\|. \end{aligned}$$

Thus (3.2) holds. □

The converse to the above claim is false in general, as can be seen by the example $\mathcal{V} = \mathbb{R}^2$ endowed with the norm $\|(x,y)\|_\infty = |x| \vee |y|$. Take

$$u = (0,0), \quad v = (1, 1/4) \quad \text{and} \quad w = (1/2, 1/4).$$

Then

$$\|u - v\|_\infty = \|u - w\|_\infty + \|w - v\|_\infty$$

but $w \notin [u,v]$.

The problem in the preceding example is that the norm is not strictly convex. We give the definition for complex vector spaces, but the same will hold for real vector spaces.

Definition 3.4. The norm $\|\cdot\|$ on the real or complex vector space $\mathcal{V}$ is called *strictly convex*, if the following hold:

$$\|u + v\| = \|u\| + \|v\| \quad \textit{and} \quad u \neq 0 \Longrightarrow \quad v = cu \text{ for some } c \geq 0.$$

Proposition 3.5. *Assume the norm $\|\cdot\|$ strictly convex. Then*

$$\|u - v\| = \|u - w\| + \|w - v\| \quad \Longleftrightarrow \quad w \in [u,v]$$

Proof. If $u = w$ the result is trivial. Assume that $u \neq w$. Since $u - v = (u - w) + (w - v)$ it follows from the definition that $w - v = c(u - w)$ for some $c \geq 0$. Then

$$w = \frac{c}{1+c}u + \frac{1}{1+c}v = u + t(v - u)$$

with ($t = 0$ corresponds to $c \longrightarrow \infty$) $t = \frac{1}{1+c} \in [0, 1]$. □

Examples of norms on the dimensional vector space $\mathbb{C}^N$ are given by (with $z = (z_1, \ldots, z_N)$ and similarly for w)

$$\|z\|_r = \begin{cases} \left(\sum_{n=1}^N m_n |z_n|^r\right)^{1/r}, & r \in [1, \infty) \\ \vee_{n=1}^N m_n |z_n|, & r = \infty \end{cases}$$

where $m_1, \ldots, m_N$ are strictly positive. They are strictly convex for $1 < r < \infty$ but not for $r \in \{1, \infty\}$, and correspond to the distances

$$D_r(z, w) = \begin{cases} \left(\sum_{n=1}^N m_n |z_n - w_n|^r\right)^{1/r}, & r \in [1, \infty) \\ \vee_{n=1}^N m_n |z_n - w_n|, & r = \infty. \end{cases}$$

These norms fall into a larger family of norms used [24] for membership functions, and which may be defined as follows. We will assume that $(X, \mathcal{A}, \sigma)$ is a measured space, with sigma-algebra $\mathcal{A}$ and positive measure σ. The measure σ has the following properties (which allows to define these norms on membership functions, since the latter take values in $[0, 1]$)

Definition 3.6. σ will be a positive measure such that $\int_X d\sigma(x) < \infty$ and with the condition:

$$\int_X |f(x)| d\sigma(x) = 0 \quad \Longrightarrow \quad f = 0, \ a.e. \tag{3.3}$$

We will say that two sets in $\mathcal{A}$ are *equivalent* (notation: $A \sim B$), if $\sigma(A \Delta B) = 0$. We have an equivalent relation since:
(1) It is reflexive since $A \Delta A = \varnothing$ and then $\sigma(A \Delta A) = 0$.
(2) It is symmetric since $A \Delta B = B \Delta A$.
(3) It is transitive. For $A, B, C \in \mathcal{N}$ assume $A \sim B$ and $B \sim C$. Then $A \sim C$ since

$$A \Delta C = (A \Delta B) \Delta (B \Delta C)$$

and

$$\sigma(A \Delta C) = \sigma((A \Delta B) \Delta (B \Delta C)) \leq \sigma(A \Delta B) + \sigma(B \Delta C) = 0.$$

Definition 3.7. We denote by $\mathcal{N}_0$ the elements of $\mathcal{A}$ equivalent to $\varnothing$ and by $\mathcal{A}_0 = \mathcal{A}/\mathcal{N}$ the space of equivalent classes.

We set, for f measurable and bounded in modulus,

$$\|f\|_r = \left(\int_X |f(x)|^r d\sigma(x)\right)^{1/r}, \quad r \in [1, \infty)$$

and

$$\|f\|_\infty = \operatorname{ess\,sup}_{x \in X} |f(x)|,$$

corresponding to distances d_r and d_∞.

The following result is of limited interest since the number t in (3.4) does not depend on x, but stresses the difference with the results presented in Sections 5 and 7.

Proposition 3.8. *Given $(X, \mathcal{A}, \sigma)$ a measured space, assume that $\mu_{\widetilde{A}}, \mu_{\widetilde{B}}$ and $\mu_{\widetilde{C}}$ are measurable membership functions, and that $\mu_{\widetilde{C}}$ is between $\mu_{\widetilde{A}}$ and $\mu_{\widetilde{C}}$ in the sense of Definition 3.1 meaning that there exists $t \in [0,1]$, independent of x, such that*

$$\mu_{\widetilde{C}}(x) = t\mu_{\widetilde{A}}(x) + (1-t)\mu_{\widetilde{B}}, \quad x \in X. \tag{3.4}$$

Then

$$d_r(\mu_{\widetilde{A}}, \mu_{\widetilde{B}}) = d_r(\mu_{\widetilde{A}}, \mu_{\widetilde{C}}) + d_r(\mu_{\widetilde{C}}, \mu_{\widetilde{B}}) \quad \forall r \in [1, \infty] \tag{3.5}$$

The converse statement is true if $r \notin \{1, \infty\}$.

Proof. The direct claim follows from Proposition 3.3. We now turn to the converse statement. Since $r \in (1, \infty)$ the norm d_r is strictly convex. Then equality in the triangle inequality means that $\mu_{\widetilde{C}}$ is in the interval defined by $\mu_{\widetilde{A}}$ and $\mu_{\widetilde{B}}$. □

4. Betweenness of sets

In preparation for the following sections we rewrite in a slightly different form some results from [9]. We first recall a definition.

Definition 4.1. Let X be some non-empty set. A set $A \in \mathcal{P}(X)$ is uniquely determined by its indicator function 1_A defined by

$$1_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A. \end{cases} \tag{4.1}$$

There is therefore in classical set theory a one-to-one correspondence between elements of $\mathcal{P}(X)$ and the set $\{0,1\}^X$ of functions from X into $\{0,1\}$. As is well known, the indicator functions of the union, intersection and symmetric difference of two sets A and B and of the complement of a set A are given by

$$\begin{aligned}
1_{A\cup B} &= 1_A + 1_B - 1_A 1_B && (4.2)\\
&= 1_A \vee 1_B, && (4.3)\\
1_{A\cap B} &= 1_A \wedge 1_B && (4.4)\\
&= 1_A 1_B, && (4.5)\\
1_{A\Delta B} &= 1_A + 1_B - 2\cdot 1_A 1_B && (4.6)\\
&= 1_A \vee 1_B - 1_A \wedge 1_B && (4.7)\\
&= (1_A - 1_B)^2, && (4.8)\\
1_{X\setminus A} &= 1 - 1_A, && (4.9)
\end{aligned}$$

where we have denoted by Δ the symmetric difference and by $X \setminus A$ the complement of the set A.

Lemma 4.2. *Let X be a set and let $A, B, C \in \mathcal{P}(X)$. Then, C is between A and B in the sense of equation (1.5) if and only if*

$$1_C = 1_{A\cap B} + 1_Z, \tag{4.10}$$

where $Z \in \mathcal{P}(X)$ is such that

$$Z \subset A\Delta B. \tag{4.11}$$

Proof. We rewrite (1.4) as (1.5), i.e.

$$1_A(x) \wedge 1_B(x) \le 1_C(x) \le 1_A(x) \vee 1_B(x) \quad \forall x \in \mathbb{R}.$$

Then there exists $t(x) \in [0,1]$ such that

$$1_C(x) = 1_A(x) \wedge 1_B(x) + t(x) \underbrace{(1_A(x) \vee 1_B(x) - 1_A(x) \wedge 1_B(x))}_{1_{A\Delta B}(x)}.$$

From the above, $t(x)$ can be chosen to belong to $\{0, 1\}$. Define a set Z via

$$1_Z(x) = t(x) \underbrace{(1_A(x) \vee 1_B(x) - 1_A(x) \wedge 1_B(x))}_{1_{A\Delta B}(x)}.$$

Thus $Z \subset A\Delta B$. The converse is clear. □

Given two sets A and B in $\mathcal{P}(\Omega)$ we note that the interval $[1_A, 1_B]$ is not made of indicator functions in general, but consists of the functions of the form

$$f_t(x) = t1_A(x) + (1-t)1_B(x) \tag{4.12}$$

when t varies in $[0, 1]$. A more interesting case is when t is allowed to vary with x:

$$f_t(x) = t(x)1_A(x) + (1 - t(x))1_B(x), \tag{4.13}$$

where now t is a function from X into $[0, 1]$. These functions are examples of membership functions, which are the main tool in fuzzy set theory. The interpretation of (4.13) in terms of norms uses the α-cuts. See Section 7 below.

For the following result, see also Restle [9], where the importance of the equality case in the triangle inequality is stressed out.

Proposition 4.3. *Let $(X, \mathcal{A}, \sigma)$ be as above with σ a measure on X satisfying the hypothesis of Definition 3.6, and let $\mathcal{A}_0$ be as in Definition 3.7. Then*

$$D_\sigma(A_0, B_0) = \int_X (1_A(x) - 1_B(x))^2 d\sigma(x) = \sigma(A\Delta B), \quad A, B \in \mathcal{A}_0$$

(where $A \in \mathcal{A}$ belongs to the equivalence class of A_0 and $B \in \mathcal{A}$ belongs to the equivalence class of B_0) is a metric on $\mathcal{A}_0$. If X has finite cardinal one can take $\mathcal{A}_0 = \mathcal{P}(X)$.

Proof. The various definitions do not depend on the chosen representative in a given equivalence class. Assume that $D_\sigma(A_0, B_0) = 0$. Then by (3.3), we have $\sigma(A\Delta B) = 0$ and thus $A_0 = B_0$. It is clear that $D_\sigma(A_0, B_0) = D_\sigma(B_0, A_0)$. We now check the triangle inequality and first note that (with $C_0 \in \mathcal{A}_0$ and C in the equivalence class C_0)

$$(1_A - 1_C)^2(x) + (1_C - 1_B)^2(x) - (1_A - 1_B)^2(x) = 2(1_A - 1_C)(x)(1_B - 1_C)(x) \tag{4.14}$$

for $x \in X$. Thus we have

(4.15)

$$D_\sigma(A_0,C_0)+D_\sigma(B_0,C_0)-D_\sigma(A_0,B_0)= \\ =\int_X \left\{(1_A-1_C)^2(x)+(1_C-1_A)^2(x)-(1_A-1_B)^2(x)\right\}d\sigma(x) \\ =2\int_X (1_A-1_C)(x)(1_B-1_C)(x)d\sigma(x).$$

But

(4.16)

$$(1_A-1_C)(x)(1_B-1_C)(x=\begin{cases}(1-1_C(x))^2=1-1_C(x), & x\in A\cap B\\ -1_C(x)(1-1_C(x))=0, & x\in B\setminus A\\ (1-1_C(x))1_C(x)=0, & x\in A\setminus B\\ 1_C(x)^2=1_C(x), & x\in X\setminus(A\cup B).\end{cases}$$

So we get

$$\int_X (1_A-1_C)(x)(1_B-1_C)(x)d\sigma(x)=\int_{A\cap B}(1-1_C(x))d\sigma(x)+\int_{X\setminus(A\cup B)}1_C(x)d\sigma(x)$$

Hence (4.15) becomes

$$D_\sigma(A_0,C_0)+D_\sigma(B_0,C_0)-D_\sigma(A_0,B_0)=\int_{A\cap B}(1-1_C(x))d\sigma(x)+\int_{X\setminus(A\cup B)}1_C(x)d\sigma(x),$$

which is non-negative. Therefore the triangle inequality holds for D_σ. □

Proposition 4.4. *In the notation of the previous proposition, C_0 is between A_0 and B_0 for the metric $D_\sigma(A_0,B_0)$ if and only if the triangle inequality is an equality:*

$$D_\sigma(A_0,B_0)=D_\sigma(A_0,C_0)+D_\sigma(C_0,B_0) \tag{4.17}$$

Proof. By the triangle inequality for D_σ and using (4.14), we have

$$0\le\int_X\left\{1_{A\Delta C}(x)+1_{C\Delta B}(x)-1_{A\Delta B}(x)\right\}d\sigma(x) \\ =\int_X\left\{(1_A-1_C)^2(x)-(1_C-1_B)^2(x)-(1_A-1_B)^2(x)\right\}d\sigma(x) \\ =2\int_X(1_A-1_C)(x)(1_B-1_C)(x)d\sigma(x).$$

Then

$$0\le\int_X\left\{1_{A\Delta C}(x)+1_{C\Delta B}(x)-_{A\Delta B}(x)\right\}d\sigma(x) \\ =2\left\{\int_{A\cap B}(1-1_C(x))d\sigma(x)+\int_{X\setminus(A\cup B)}1_C(x)\right\}d\sigma(x)$$

Thus (4.17) holds if and only if

$$\int_{A\cap B}(1-1_C(x))d\sigma(x)=\int_{X\setminus(A\cup B)}1_C(x)d\sigma(x)=0$$

that is, if and only if

$$1_C(x) = 1, \quad x \in A \cap B \quad \text{and} \quad 1_C(x) = 0, \quad x \notin A \cup B,$$

i.e. if and only if (1.4) is in force. □

$\sqrt{D_\sigma}$ is also a metric on $\mathcal{A}$ (maybe more natural a priori since it arises from a positive definite kernel), but we have:

Proposition 4.5. *Let $A_0, B_0, C_0 \in \mathcal{A}_0$. It holds that*

$$\sqrt{D_\sigma(A_0, B_0)} = \sqrt{D_\sigma(A_0, C_0)} + \sqrt{D_\sigma(C_0, B_0)} \tag{4.18}$$

if and only if $C_0 = A_0$ or $C_0 = B_0$.

Proof. Assume that (4.18) is in force. Taking square and taking into account (4.14) we obtain, with A, B, C being in the equivalence classes of A_0, B_0 and C_0 respectively

$$-2\int_X (1_A - 1_C)(x)(1_B - 1_C)(x)d\sigma(x) = 2\sqrt{D_\sigma(A_0, C_0)}\sqrt{D_\sigma(C_0, B_0)}. \tag{4.19}$$

Then we are in the equality case in the Cauchy-Schwarz inequality. If $C = A$, then there is nothing to prove. Assume that $C_0 \neq A_0$. Then there exists $u \in \mathbb{C}$ such that

$$(1_B - 1_C) = u(1_C - 1_A), \quad \sigma \text{ a.e.} \tag{4.20}$$

Plugging this into (4.19), we obtain

$$u\int_X (1_C(x) - 1_A(x))^2 d\sigma(x) = |u| \cdot \int_X (1_C(x) - 1_A(x))^2 d\sigma(x).$$

Since $A \neq C$ it follows that $u \geq 0$.

If $u = 0$ in (4.20), then we have $B = C$. Thus $B_0 = C_0$. We now show by contradiction that we cannot have $u \neq 0$ since $C \neq A$. Assume thus $u \neq 0$ (and so $u > 0$) and first suppose that there is $x \in C \setminus A$. Then (4.20) becomes

$$(1_B - 1) = u.$$

The left handside of this equality is less or equal to 0 while the right handside is strictly positive, which is impossible. Assume now that there is $x \in A \setminus C$ Then (4.20) becomes

$$1_B = -u$$

which is impossible for the same reason as above. □

5. Fuzzy set theory

In a way similar to information theory, which originates in 1948 with Shannon's paper [25], one can pinpoint the origin of fuzzy set theory and logic with the papers of Zadeh [11], but it is good to mention the earlier works on multi-valued logic of Lukasiewicz [26]. For the convenience of the reader we review some definitions from fuzzy set theory, and send the reader to the books [10, 27, 28, 29, 30] for further information.

The set of indicator functions is $\{0, 1\}^X$, and is therefore included in the set of membership functions (See Definition 1.1 for the latter). Let $N \in \mathbb{N}$. We note that to any function from $[0, 1]^N$ into $[0, 1]$ one can define a map which to N membership functions associates a new membership function.

Each of the functions (4.2)-(4.7) (and to a certain extent also (4.9)) have numerous possible extensions in the setting of membership functions. This degree of freedom is one of the main strengths of fuzzy set theory. As a first example, consider the intersection, with indicator function $1_A 1_B$. When 1_A and 1_B are replaced by membership functions $\mu_{\widetilde{A}}$ and $\mu_{\widetilde{B}}$, two different extensions of intersection of classical sets will be given by $\mu_{\widetilde{A}} \vee \mu_{\widetilde{B}}$ and $\mu_{\widetilde{A}}\mu_{\widetilde{B}}$.

The maximum and minimum of two membership functions $\mu_{\widetilde{A}}$ and $\mu_{\widetilde{B}}$ are also membership functions, corresponding respectively to the union $\widetilde{A}\cup\widetilde{B}$ and intersection $\widetilde{A}\cap\widetilde{B}$ of the fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ (See [10, §3.1 p. 30]).

The product $\mu_{\widetilde{A}}\mu_{\widetilde{B}}$ is also a membership functions, corresponding to a fuzzy set called algebraic product of the fuzzy sets $\widetilde{A}$ and $\widetilde{B}$; see [10, §3.3 p. 33]).

For $\mu_{\widetilde{A}}$ a membership function, $1-\mu_{\widetilde{A}}$ is still a membership function, corresponding to a fuzzy set called the fuzzy complement of $\widetilde{A}$, and denote by $cl(\widetilde{A})$.

More generally, one can take functions with values in a lattice; this was already done by Zadeh's student Goguen, see [31], and later also developped by Atanassov in his theory of intuitionistic sets; see [32]. An intuitionistic fuzzy set defined on a set X is defined by two functions from X into $[0,1]$, respectively called membership function and non-membership function. In the second part of this paper (Sections 8-11) we will consider the lattice of hyperbolic numbers.

6. Betweenness in the fuzzy case

Definition 6.1. Let $\mu_{\widetilde{A}}, \mu_{\widetilde{B}}$ and $\mu_{\widetilde{C}}$ be membership functions. We say that $\mu_{\widetilde{C}}$ is *pointwise between* $\mu_{\widetilde{A}}$ *and* $\mu_{\widetilde{B}}$, if

$$\mu_{\widetilde{A}}(x) \wedge \mu_{\widetilde{B}}(x) \le \mu_{\widetilde{C}}(x) \le \mu_{\widetilde{A}}(x) \vee \mu_{\widetilde{B}}(x) \quad \forall x \in X. \tag{6.1}$$

In other words, for every $x \in X$, $\mu_{\widetilde{C}}(x)$ belongs to the interval determined by $\mu_{\widetilde{A}}(x)$ and $\mu_{\widetilde{B}}(x)$.

As it should be this definition reduces to (1.4) in the crisp case since then we have

$$1_{A\cap B}(x) = \mu_{\widetilde{A}}(x) \wedge \mu_{\widetilde{B}}(x) \quad \text{and} \quad 1_{A\cup B}(x) = \mu_{\widetilde{A}}(x) \vee \mu_{\widetilde{B}}(x) \quad \forall x \in \mathbb{R}.$$

The counterpart of Lemma 4.2 is as follows:

Proposition 6.2. *Let* $\mu_{\widetilde{A}}, \mu_{\widetilde{B}}$ *and* $\mu_{\widetilde{C}}$ *be membership functions. Then,* $\mu_{\widetilde{C}}$ *is pointwise between* $\mu_{\widetilde{A}}$ *and* $\mu_{\widetilde{B}}$ *if and only if*

$$\mu_{\widetilde{C}}(x) = \mu_{\widetilde{A}}(x) \wedge \mu_{\widetilde{B}}(x) + \mu_{\widetilde{Z}}(x), \tag{6.2}$$

where $\mu_{\widetilde{Z}}$ *is a membership function satisfying*

$$\mu_{\widetilde{Z}}(x) \le \mu_{\widetilde{A}}(x) \vee \mu_{\widetilde{B}}(x) - \mu_{\widetilde{A}}(x) \wedge \mu_{\widetilde{B}}(x). \tag{6.3}$$

Proof. Assume first that (6.2) and (6.3) are in force. (6.2) implies that

$$\mu_{\widetilde{C}}(x) \ge \mu_{\widetilde{A}}(x) \wedge \mu_{\widetilde{B}}(x),$$

and (6.2) and (6.3) together lead to

$$\begin{aligned}\mu_{\widetilde{C}}(x) &\leq \mu_{\widetilde{A}}(x) \wedge \mu_{\widetilde{B}}(x) + \mu_{\widetilde{A}}(x) \vee \mu_{\widetilde{B}}(x) - \mu_{\widetilde{A}}(x) \wedge \mu_{\widetilde{B}}(x) \\ &= \mu_{\widetilde{A}}(x) \vee \mu_{\widetilde{B}}(x).\end{aligned}$$

Conversely, assume that (6.1) holds. The formula

$$\mu_{\widetilde{Z}}(x) = \mu_{\widetilde{C}}(x) - \mu_{\widetilde{A}}(x) \wedge \mu_{\widetilde{B}}(x)$$

defines a membership function which answers the question. □

Remark 6.3. In the crisp case, (6.2)-(6.3) reduce to (4.10)-(4.11).

7. Betweenness in the fuzzy case with α-cuts

Metrics between membership functions using α-cuts have been defined in [33]. For $\alpha \in [0, 1]$ we consider the strong α-cuts A'_α associated to the membership function $\mu_{\widetilde{A}}$, defined by

$$A'_\alpha = \mu_{\widetilde{A}}^{-1}(\alpha, 1], \quad \alpha \in [0, 1]. \tag{7.1}$$

Note than one also defines α-cuts

$$A_\alpha = \mu_{\widetilde{A}}^{-1}[\alpha, 1], \quad \alpha \in [0, 1]. \tag{7.2}$$

See e.g. [30, p. 14]. The arguments in this section will not hold with the latter definition; strict inequalities are needed.

Definition 7.1. Let $\mu_{\widetilde{A}}, \mu_{\widetilde{B}}$ and $\nu_{\widetilde{C}}$ be membership functions from X to $[0, 1]$. We say that $\mu_{\widetilde{C}}$ is *α-between $\mu_{\widetilde{A}}$ and $\mu_{\widetilde{B}}$*, if $\mu_{\widetilde{C}}^{-1}(\alpha, 1]$ is between $\mu_{\widetilde{A}}^{-1}(\alpha, 1]$ and $\mu_{\widetilde{B}}^{-1}(\alpha, 1]$ for every $\alpha \in [0, 1]$.

Theorem 7.2. *$\mu_{\widetilde{C}}$ is α-between $\mu_{\widetilde{A}}$ and $\mu_{\widetilde{B}}$ if and only if $\mu_{\widetilde{C}}$ is pointwise between $\mu_{\widetilde{A}}$ and $\mu_{\widetilde{B}}$.*

Proof. We first assume that $\mu_{\widetilde{C}}$ is α-between $\mu_{\widetilde{A}}$ and $\mu_{\widetilde{B}}$. Let $x \in X$ and let $\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)$ and $\mu_{\widetilde{C}}(x)$ be the corresponding values of the membership functions. Since $\mu_{\widetilde{A}}$ and $\mu_{\widetilde{B}}$ play a symmetric role we can assume without loss of generality that

$$\mu_{\widetilde{A}}(x) \leq \mu_{\widetilde{B}}(x). \tag{7.3}$$

We want to show that (6.1) holds for all $x \in X$, i.e. taking into account (7.3), that

$$\mu_{\widetilde{A}}(x) \leq \mu_{\widetilde{C}}(x) \leq \mu_{\widetilde{B}}(x). \tag{7.4}$$

Equivalently, we have to show that the following cannot hold:

$$\mu_{\widetilde{C}}(x) < \mu_{\widetilde{A}}(x) \tag{7.5}$$

or

$$\mu_{\widetilde{B}}(x) < \mu_{\widetilde{C}}(x). \tag{7.6}$$

Assume first by contradiction that (7.5) holds. Then $x \in \mu_{\widetilde{A}}^{-1}(\mu_{\widetilde{C}}(x), 1]$. By (7.3), we also have $x \in \mu_{\widetilde{B}}^{-1}(\mu_{\widetilde{C}}(x), 1]$. Thus

$$\mu_{\widetilde{A}}^{-1}(\mu_{\widetilde{C}}(x), 1] \cap \mu_{\widetilde{B}}^{-1}(\mu_{\widetilde{C}}(x), 1] \neq \varnothing$$

since x belongs to the intersection. But

$$\mu_{\widetilde{C}}^{-1}(\mu_{\widetilde{C}}(x),1]=\left\{y\in X\ :\ \mu_{\widetilde{C}}(x)<\mu_{\widetilde{C}}(y)\leq 1\right\}$$

and so $x\notin\mu_{\widetilde{C}}^{-1}(\mu_{\widetilde{C}}(x),1]$, leading to a contradiction since, with $\alpha=\mu_{\widetilde{C}}(x)$ the hypothesis of α-betweenness of $\mu_{\widetilde{C}}$ between $\mu_{\widetilde{A}}$ and $\mu_{\widetilde{B}}$ gives

$$\underbrace{\mu_{\widetilde{A}}^{-1}(\mu_{\widetilde{C}}(x),1]\cap\mu_{\widetilde{B}}^{-1}(\mu_{\widetilde{C}}(x),1]}_{\text{contains } x}\subset\underbrace{\mu_{\widetilde{C}}^{-1}(\mu_{\widetilde{C}}(x),1].}_{\text{does not contain } x}$$

Assume now by contradiction that (7.6) holds. Then $x\in\mu_{\widetilde{C}}^{-1}(\mu_{\widetilde{B}}(x),1]$ but $x\notin\mu_{\widetilde{B}}^{-1}(\mu_{\widetilde{B}}(x),1]$. The α-betweenness with $\alpha=\mu_{\widetilde{B}(x)}$ gives the the decomposition

$$\mu_{\widetilde{C}}^{-1}(\mu_{\widetilde{B}}(x),1]=\underbrace{\left(\mu_{\widetilde{A}}^{-1}(\mu_{\widetilde{B}}(x),1]\cap\mu_{\widetilde{B}}^{-1}(\mu_{\widetilde{B}}(x),1]\right)}_{\varnothing\text{ since } x\notin\mu_{\widetilde{B}}^{-1}(\mu_{\widetilde{B}}(x),1]}\cup Z_x$$

with

$$Z_x\subset\left(\mu_{\widetilde{A}}^{-1}(\mu_{\widetilde{B}}(x),1]\setminus\mu_{\widetilde{B}}^{-1}(\mu_{\widetilde{B}}(x),1]\right)\cup\underbrace{\left(\mu_{\widetilde{B}}^{-1}(\mu_{\widetilde{B}}(x),1]\setminus\mu_{\widetilde{A}}^{-1}(\mu_{\widetilde{B}}(x),1]\right)}_{\varnothing\text{ since } x\notin\mu_{\widetilde{B}}^{-1}(\mu_{\widetilde{B}}(x),1]}.$$

See Lemma 4.2 and equation (4.11). Since $x\notin\mu_{\widetilde{B}}^{-1}(\mu_{\widetilde{B}}(x),1]$ we have that $x\in Z_x$ and in particular $x\in\mu_{\widetilde{A}}^{-1}(\mu_{\widetilde{B}}(x),1]$, so that

$$\mu_{\widetilde{A}}(x)>\mu_{\widetilde{B}}(x),\tag{7.7}$$

contradicting (7.3).

Conversely, assume that $\mu_{\widetilde{C}}$ is pointwise between $\mu_{\widetilde{A}}$ and $\mu_{\widetilde{B}}$. Then for every $x\in X$,

$$\mu_{\widetilde{C}}(x)\in[\mu_{\widetilde{A}}(x)\wedge\mu_{\widetilde{B}}(x),\mu_{\widetilde{A}}(x)\vee\mu_{\widetilde{B}}(x)]\tag{7.8}$$

is in force. We want to show that, for every $\alpha\in[0,1]$

$$\mu_{\widetilde{A}}^{-1}(\alpha,1]\cap\mu_{\widetilde{B}}^{-1}(\alpha,1]\subset\mu_{\widetilde{C}}^{-1}(\alpha,1]\subset\mu_{\widetilde{A}}^{-1}(\alpha,1]\cup\mu_{\widetilde{B}}^{-1}(\alpha,1].\tag{7.9}$$

We divide this part of the proof in a number of steps.

STEP 1: *If there is no x such that $\mu_{\widetilde{C}}(x)>\alpha$, both inclusions in (7.9) are satisfied.*

The second inclusion in (7.9) is now trivial. We show that the first one holds (and reduces to $\varnothing=\varnothing$). By hypothesis,

$$\mu_{\widetilde{C}}(x)\leq\alpha\tag{7.10}$$

for all $x\in X$. Assume by contradiction that there is $y\in\mu_{\widetilde{A}}^{-1}(\alpha,1]\cap\mu_{\widetilde{B}}^{-1}(\alpha,1]$. Then

$$\alpha<\mu_{\widetilde{A}}(y)\leq 1\quad\text{and}\quad\alpha<\mu_{\widetilde{B}}(y)\leq 1.$$

In particular

$$\alpha<\mu_{\widetilde{A}}(y)\wedge\mu_{\widetilde{B}}(y).\tag{7.11}$$

By the hypothesis on pointwise betweenness

$$\mu_{\widetilde{A}}(y) \wedge \mu_{\widetilde{B}}(y) \le \mu_{\widetilde{C}}(y) \tag{7.12}$$

Equations (7.10), (7.11) and (7.12) lead to

$$\alpha < \mu_{\widetilde{A}}(y) \wedge \mu_{\widetilde{B}}(y) \le \mu_{\widetilde{C}}(y) \le \alpha,$$

which cannot be.

STEP 2: *The first inclusion in* (7.9) *holds.*

If $\mu_{\widetilde{A}}^{-1}(\alpha,1] \cap \mu_{\widetilde{B}}^{-1}(\alpha,1] = \varnothing$ the first inclusion is trivially met. Assume now that there is $x \in \mu_{\widetilde{A}}^{-1}(\alpha,1] \cap \mu_{\widetilde{B}}^{-1}(\alpha,1]$. Then x is such that $\mu_{\widetilde{A}}(x) \in (\alpha,1]$ and $\mu_{\widetilde{B}}(x) \in (\alpha,1]$. Thus

$$\alpha < \mu_{\widetilde{A}}(x) \le 1 \quad \text{and} \quad \alpha < \mu_{\widetilde{B}}(x) \le 1.$$

From $\mu_{\widetilde{A}}(x) \wedge \mu_{\widetilde{B}}(x) \le \mu_{\widetilde{C}}(x)$ we have that $x \in \mu_{\widetilde{C}}^{-1}(\alpha,1]$. So

$$\mu_{\widetilde{A}}^{-1}(\alpha,1] \cap \mu_{\widetilde{B}}^{-1}(\alpha,1] \subset \mu_{\widetilde{C}}^{-1}(\alpha,1].$$

STEP 3: *The second inclusion in* (7.9) *holds.*

By Step 1 we may assume that $\mu_{\widetilde{C}}^{-1}(\alpha,1] \neq \varnothing$. Let thus x be such that $\mu_{\widetilde{C}}(x) > \alpha$. Since $\mu_{\widetilde{A}}(x) \vee \mu_{\widetilde{B}}(x) \ge \mu_{\widetilde{C}}(x)$. we have

$$\mu_{\widetilde{C}}^{-1}(\alpha,1] \subset \mu_{\widetilde{A}}^{-1}(\alpha,1] \cup \mu_{\widetilde{B}}^{-1}(\alpha,1].$$

Then $x \in \mu_{\widetilde{A}}^{-1}(\alpha,1] \cup \mu_{\widetilde{B}}^{-1}(\alpha,1]$ and thus $\mu_{\widetilde{C}}(\alpha,1]$ is between $\mu_{\widetilde{A}}^{-1}(\alpha,1]$ and $\mu_{\widetilde{B}}^{-1}(\alpha,1]$. □

Let now σ be a measure on X satisfying the properties of Definition 3.6 and let η be a strictly positive measure on $[0,1]$ We define (assuming the integrals well defined)

$$D(\mu_{\widetilde{A}}, \mu_{\widetilde{B}}) = \int_0^1 \left(\int_X \left(1_{\mu_{\widetilde{A}}^{-1}(\alpha,1] \Delta \mu_{\widetilde{B}}^{-1}(\alpha,1]}(x) \right) d\sigma(x) \right) d\eta(\alpha). \tag{7.13}$$

Theorem 7.3. *Assuming the integral well defined,* (7.13) *defines a metric, and* $\mu_{\widetilde{C}}$ *is pointwise between* $\mu_{\widetilde{A}}$ *and* $\mu_{\widetilde{B}}$ *if and only if the equality holds in the triangle inequality for this metric.*

Proof. By Proposition 4.3, we have that for every $\alpha \in [0,1]$ the formula

$$\int_X 1_{A \Delta B}(x) d\sigma(x)$$

defines a metric on $\mathcal{P}(X)$. Then (7.13) is an integral of metrics and Thus a metric.

To prove the claim in the theorem we go along the lines of Proposition 4.4, using (4.14) with A replaced by $\mu_{\widetilde{A}}^{-1}(\alpha,1]$ and similarly for B and C. We can write:

$$
\begin{aligned}
&0 \le D(\mu_{\widetilde{A}},\mu_{\widetilde{C}}) + D(\mu_{\widetilde{C}},\mu_{\widetilde{B}}) - D(\mu_{\widetilde{A}},\mu_{\widetilde{B}}) = \\
&= 2\int_0^1 \left(\int_X \left(1_{\mu_{\widetilde{A}}^{-1}(\alpha,1]}(x) - 1_{\mu_{\widetilde{C}}^{-1}(\alpha,1]}(x)\right)\left(1_{\mu_{\widetilde{B}}^{-1}(\alpha,1]}(x) - 1_{\mu_{\widetilde{C}}^{-1}(\alpha,1]}(x)\right) d\eta(x)\right) d\sigma(\alpha)\\
&= 2\int_0^1 \left(\left(\int_{\mu_{\widetilde{A}}^{-1}(\alpha,1]\cap\mu_{\widetilde{B}}^{-1}(\alpha,1]} \left(1 - 1_{\mu_{\widetilde{C}}^{-1}(\alpha,1]}(x)\right) d\eta(x) + \right.\right.\\
&\qquad \left.\left. + \int_{X\setminus(\mu_{\widetilde{A}}^{-1}(\alpha,1]\cup\mu_{\widetilde{B}}^{-1}(\alpha,1])} 1_{\mu_{\widetilde{C}}^{-1}(\alpha,1]}(x) d\eta(x)\right)\right) d\sigma(\alpha)
\end{aligned}
$$

By the assumed properties on $d\sigma(\alpha)$ we have therefore equality in the triangle inequality if and only if

$$
\int_{\mu_{\widetilde{A}}^{-1}(\alpha,1]\cap\mu_{\widetilde{B}}^{-1}(\alpha,1]} \left(1 - 1_{\mu_{\widetilde{C}}^{-1}(\alpha,1]}(x)\right) d\eta(x) = \int_{X\setminus(\mu_{\widetilde{A}}^{-1}(\alpha,1]\cup\mu_{\widetilde{B}}^{-1}(\alpha,1])} 1_{\mu_{\widetilde{C}}^{-1}(\alpha,1]}(x) d\eta(x) = 0
$$

and the end of the proof is as in the proof of Proposition 4.4. □

8. The hyperbolic numbers

Complex numbers can be constructed as matrices of the form $\begin{pmatrix} a & -b \\ b & a\end{pmatrix}$, where $a,b\in\mathbb{R}$ and can be viewed (when $(a,b)\neq(0,0)$) as composition of an homothety and a rotation in the plane:

$$
\rho\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}
$$

Hyperbolic number in turn are symmetric matrices of the form

$$
\begin{pmatrix} a & b\\ b & a\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix} a+b & 0\\ 0 & a-b\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \tag{8.1}
$$

and can be seen when $a^2-b^2\neq 0$ as composition of an homothety and an hyperbolic rotation

$$
\rho\begin{pmatrix}\cosh\theta & \sinh\theta\\ \sinh\theta & \cosh\theta\end{pmatrix}.
$$

Then hyperbolic numbers from a family of pairwise commuting matrices; we refer to [34, 35] for more information on these numbers.

It will be convenient to set

$$
U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}. \tag{8.2}
$$

Note that

$$
U = U^t \quad\text{and}\quad U^2 = I_2.
$$

We have

$$
\begin{pmatrix} a & -b\\ -b & a\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix} a-b & 0\\ 0 & a+b\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \tag{8.3}
$$

and

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} a & -b \\ -b & a \end{pmatrix} = (a^2-b^2)I_2 \tag{8.4}$$

Not every non-zero hyperbolic number is invertible but the formula

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}^{-1} = \frac{1}{a^2-b^2}\begin{pmatrix} a & -b \\ -b & a \end{pmatrix}, \quad a^2-b^2\neq 0, \tag{8.5}$$

shows in particular that the set of hyperbolic numbers for which $a^2-b^2\neq 0$ form a multiplicative Abelian group of matrices, a subgroup of which consists of the matrices for which $a^2-b^2=1$.

Thus:

Lemma 8.1. *The hyperbolic number satisfies*

$$0\le \begin{pmatrix} a & b \\ b & a \end{pmatrix} \le \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8.6}$$

if and only if it holds that

$$0\le a+b\le 1 \quad \textit{and} \quad 0\le a-b\le 1. \tag{8.7}$$

Proof. This is a direct consequence of (8.1). □

Note that in the (a,b) plane the set (8.7) is the square with vertices

$$(0,0),\ (1/2,1/2),\ (-1/2,-1/2) \quad \text{and} \quad (1,0).$$

Definition 8.2. We denote by $\mathbb{D}$ the set of hyperbolic numbers satisfying (8.7).

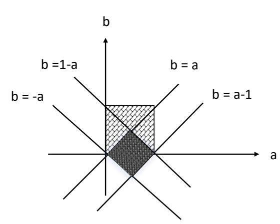

FIGURE 1. The set $\mathbb{D}$

For $z=\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ and $w=\begin{pmatrix} c & d \\ d & c \end{pmatrix}$ in $\mathbb{H}$, we define

(8.8)

$$\begin{split} z\vee w &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} (a+b)\vee(c+d) & 0 \\ 0 & (a-b)\vee(c-d) \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} (a+b)\vee(c+d)+(a-b)\vee(c-d) & (a+b)\vee(c+d)-(a-b)\vee(c-d) \\ (a+b)\vee(c+d)-(a-b)\vee(c-d) & (a+b)\vee(c+d)+(a-b)\vee(c-d) \end{pmatrix} \end{split}$$

and

(8.9)

$$z \wedge w = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}(a+b)\wedge(c+d) & 0\\ 0 & (a-b)\wedge(c-d)\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix}(a+b)\wedge(c+d)+(a-b)\wedge(c-d) & (a+b)\wedge(c+d)-(a-b)\wedge(c-d)\\ (a+b)\wedge(c+d)-(a-b)\wedge(c-d) & (a+b)\wedge(c+d)+(a-b)\wedge(c-d)\end{pmatrix}.$$

Proposition 8.3. *The set $\mathbb{H}$ with the above functions $\vee$ and $\wedge$ is a lattice when endowed with the partial order of matrices.*

Proof. It holds that

$$z \wedge w \le z \le z \vee w \quad \text{and} \quad z \wedge w \le w \le z \vee w. \tag{8.10}$$

We now discuss the uniqueness of the functions $\vee$ and $\wedge$. Given $z, w \in \mathbb{D}$ we consider positive hyperbolic numbers m and M such that

$$m \le z \le M \quad \text{and} \quad m \le w \le M \tag{8.11}$$

We note that m and M are not unique, and two hyperbolic numbers m_1 and m_2 satisfying (8.11) need not be comparable. But any m and M which satisfy (8.11) will also satisfy

$$m \le z \wedge w \quad \text{and} \quad z \vee w \le M. \tag{8.12}$$

□

As a corollary:

Corollary 8.4. *In the above notation, $z \wedge w$ and $z \vee w$ are uniquely determined to be respectively the largest and smallest hyperbolic numbers satisfying* (8.12)*.*

We now define the counterpart of an interval in the hyperbolic setting. Given two elements $z, w \in \mathbb{H}$, the characterization via (3.1) is not the one to consider here. Indeed the set

$$\{c(t) = z \wedge v + t(z \vee w - z \wedge v),\ t \in [0,1]\}$$

need not contain z or w, as illustrated by the following example. Take

$$z = U\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}U, \quad w = U\begin{pmatrix}3 & 0\\ 0 & 0\end{pmatrix}U. \tag{8.13}$$

Then

$$z \wedge w = U\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}U, \quad z \vee w = U\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}U. \tag{8.14}$$

Thus the interval

$$[z \wedge w, z \vee w] = \left\{U\begin{pmatrix}1+2t & 0\\ 0 & 2t\end{pmatrix}U,\ t \in [0,1]\right\}$$

does not contain z or w.

Recall now that $\mathbb{D}$ was defined by condition (8.7) and denotes the set of positive hyperbolic numbers less or equal to I_2.

Definition 8.5. Let $z, w \in \mathbb{H}$. We define the interval

$$[z, w]_H = \{c(\tau) = z \wedge w + \tau\,(z \vee w - z \wedge w)\,,\ \tau \in \mathbb{D}\}\,. \tag{8.15}$$

Proposition 8.6. $[z, w]_H$ *can be characterized as:*

$$[z, w]_H = \{v \in H\ :\ z \wedge w \le v \le z \vee w\}$$

Proof. Let

$$\tau = U \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} U, \quad t_1, t_2 \in [0, 1]$$

$$z = U \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} U, \quad w = U \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} U, \quad \text{and} \quad v = U \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} U.$$

We can write

(8.16)

$$c(\tau) = U \begin{pmatrix} \underbrace{\lambda_1 \wedge \mu_1 + t_1(\lambda_1 \vee \mu_1 - \lambda_1 \wedge \mu_1)}_{v_1(t_1)} & 0 \\ 0 & \underbrace{\lambda_2 \wedge \mu_2 + t_2(\lambda_2 \vee \mu_2 - \lambda_2 \wedge \mu_2)}_{v_2(t_2)} \end{pmatrix} U.$$

But, for $j = 1, 2$ and as t_j varies from 0 to 1 we have that $v_j(t_j)$ varies from $\lambda_j \wedge \mu_j$ to $\lambda_j \vee \mu_j$. Hence, the representation (8.16) for $c(\tau)$ is equivalent to

$$z \wedge w \le c(\tau) \le z \vee w.$$

□

Definition 8.7. Let $z, w, u \in \mathbb{H}$. We say that u *is between* z *and* w, if $u \in [z, w]_{\mathbb{H}}$, that is

$$z \wedge w \le u \le z \vee w.$$

We note that the notion if betweenness is not transitive. Restle already had examples of lack of transitivity for sets.

Example 8.8. Take z and w as in (8.13), with minimum and maximum as in (8.14) and

$$u = U \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} U, \quad v = U \begin{pmatrix} 5 & 0 \\ 0 & 1/2 \end{pmatrix} U.$$

Then

$$u \wedge v = U \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} U, \quad u \vee v = U \begin{pmatrix} 5 & 0 \\ 0 & 1/2 \end{pmatrix} U,$$

and

$$z \wedge v = U \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} U, \quad z \vee v = U \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} U$$

Thus $u \in [z, w]_H$, $w \in [u, v]_H$ but $u \notin [z, v]_H$.

To conclude this section we note that we can write

$$z = (a+b)P + (a-b)Q, \tag{8.17}$$

where P and Q denote the orthogonal projections

$$P = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} \quad \text{and} \quad Q = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \end{pmatrix}. \tag{8.18}$$

We further note that

$$P^2 = P, \quad Q^2 = Q, \tag{8.19}$$

$$PQ = QP = 0 \tag{8.20}$$

and

$$P + Q = I_2, \tag{8.21}$$

and that

$$P = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \tag{8.22}$$

and

$$Q = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right). \tag{8.23}$$

Representation (8.17) is called the idempotent representation of the hyperbolic number. In this work we chose to write hyperbolic numbers as matrices; one could also use the more traditional notation

$$z = a + b\mathbf{k}$$

where $\mathbf{k} \notin \mathbb{R}$ satisfies $\mathbf{k}^2 = 1$ (in the matrix notation, we have $\mathbf{k} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$).

9. $\mathbb{D}$-valued membership functions

Definition 9.1. Let X be a set. An hyperbolic-valued membership function on X is a $\mathbb{D}$-valued map, i.e. a $\mathbb{H}$-valued map, say M, satisfying

$$0 \le M(x) \le I_2, \quad x \in X. \tag{9.1}$$

Theorem 9.2. *$M(x)$ is an hyperbolic-valued membership function if and only if there exist two membership functions $\mu_{\widetilde{A_1}}$ and $\mu_{\widetilde{A_2}}$ corresponding to the fuzzy sets $\widetilde{A_1}$ and $\widetilde{A_2}$ respectively such that*

$$M(x) = \frac{1}{2}\begin{pmatrix} \mu_{\widetilde{A_1}}(x) + \mu_{\widetilde{A_2}}(x) & \mu_{\widetilde{A_1}}(x) - \mu_{\widetilde{A_2}}(x) \\ \mu_{\widetilde{A_1}}(x) - \mu_{\widetilde{A_2}}(x) & \mu_{\widetilde{A_1}}(x) + \mu_{\widetilde{A_2}}(x) \end{pmatrix}. \tag{9.2}$$

Proof. Following (8.1), we write

(9.3)
$$M(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & a(x) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a(x)+b(x) & 0 \\ 0 & a(x)-b(x) \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

By (8.7), both $a(x)+b(x)$ and $a(x)-b(x)$ take values in $[0,1]$ and then are (classical) membership functions, corresponding to fuzzy sets say $\widetilde{A_1}$ and $\widetilde{A_2}$:

$$a(x)+b(x)=\mu_{\widetilde{A_1}}(x)\quad\text{and}\quad a(x)-b(x)=\mu_{\widetilde{A_2}}(x).$$

Thus

$$M(x)=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}\mu_{\widetilde{A_1}}(x)&0\\0&\mu_{\widetilde{A_2}}(x)\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}. \tag{9.4}$$

Formula (9.2) follows. □

We will use the notation

$$M(x)=M_{\widetilde{A_1},\widetilde{A_2}}(x)$$

and denote the fuzzy set as the pair $(\widetilde{A_1},\widetilde{A_2})$. One has

$$(\mu_{\widetilde{A_1}}\wedge\mu_{\widetilde{A_2}})I_2\le M_{\widetilde{A_1},\widetilde{A_2}}(x)\le(\mu_{\widetilde{A_1}}\vee\mu_{\widetilde{A_2}})I_2 \tag{9.5}$$

It follows from (9.5) that $M_{\widetilde{A_1},\widetilde{A_2}}$ defines a set "between" the intersection and the union of the two fuzzy sets $\mu_{\widetilde{A_1}}\wedge\mu_{\widetilde{A_2}}$ and $\mu_{\widetilde{A_1}}\vee\mu_{\widetilde{A_2}}$.

We also note that we can rewrite $M_{\widetilde{A_1},\widetilde{A_2}}(x)$ as the idempotent representation

$$M_{\widetilde{A_1},\widetilde{A_2}}(x)=\mu_{\widetilde{A_1}}(x)P+\mu_{\widetilde{A_2}}(x)Q, \tag{9.6}$$

where P and Q are as in (8.18).

Definition 9.3. Let $\alpha=\begin{pmatrix}\alpha_1&\alpha_2\\\alpha_2&\alpha_1\end{pmatrix}$ be a positive hyperbolic number less or equal to I_2. We define the α-*cut* of the hyperbolic fuzzy set $M_{\widetilde{A_1},\widetilde{A_2}}$ to be

$$\left\{x\in X\ ;M_{\widetilde{A_1},\widetilde{A_2}}(x)>\alpha\right\} \tag{9.7}$$

By (8.1) we see that (9.7) is equivalent to

$$\mu_{\widetilde{A_1}}(x)\quad>\quad\alpha_1+\alpha_2 \tag{9.8}$$

$$\mu_{\widetilde{A_2}}(x)\quad>\quad\alpha_1-\alpha_2, \tag{9.9}$$

corresponding to the (possibly empty) α-cuts $(\widetilde{A_1})_{\alpha_1+\alpha_2}$ and α-cuts $(\widetilde{A_2})_{\alpha_1-\alpha_2}$.

Remark 9.4. When $\mu_{\widetilde{A_1}}=1_{A_1}$ and $\mu_{\widetilde{A_2}}=1_{A_2}$ for some subsets A_1 and A_2 of X we have

$$M_{\widetilde{A_1},\widetilde{A_2}}(x)=1_{A_1}(x)P+1_{A_2}(x)Q$$

and

$$1_{A_1\cap A_2}I_2\le M_{\widetilde{A_1},\widetilde{A_2}}(x)\le 1_{A_1\cup A_2}I_2$$

When

$$\mu_{\widetilde{A_1}}(x)\ge\mu_{\widetilde{A_2}}(x)\quad\forall x\in X,$$

both the functions

$$\mu_{ATA}(x)\quad=\quad\frac{\mu_{\widetilde{A_1}}(x)+\mu_{\widetilde{A_2}}(x)}{2} \tag{9.10}$$

$$\nu_{ATA}(x)\quad=\quad\frac{\mu_{\widetilde{A_1}}(x)-\mu_{\widetilde{A_2}}(x)}{2} \tag{9.11}$$

are membership functions, and such that

$$\mu_{ATA}(x) + \nu_{ATA}(x) \le 1,$$

and define an Atanassov intuitionistic fuzzy set.

Definition 9.5. The hyperbolic fuzzy set is called an *Atanassov hyperbolic fuzzy set*, if

$$\mu_{\widetilde{A_1}}(x) \ge \mu_{\widetilde{A_2}}(x) \quad \forall x \in X$$

Proposition 9.6. *The product of two Atanassov hyperbolic fuzzy sets is an Atanassov hyperbolic fuzzy set.*

Proof. Let $M_{\widetilde{A_1},\widetilde{A_2}}(x)$ and $M_{\widetilde{B_1},\widetilde{B_2}}(x)$ be the two Atanassov hyperbolic fuzzy set. Then

$$\mu_{\widetilde{A_1}}(x) \ge \mu_{\widetilde{A_2}}(x) \quad \text{and} \quad \mu_{\widetilde{B_1}}(x) \ge \mu_{\widetilde{B_2}}(x) \quad \forall x \in X.$$

Thus

$$\mu_{\widetilde{A_1}}(x)\mu_{\widetilde{B_1}}(x) \ge \mu_{\widetilde{A_2}}(x)\mu_{\widetilde{B_2}}(x) \quad \forall x \in X.$$

So the answer. □

10. Properties of hyperbolic membership functions

In this section we consider the counterparts in the hyperbolic setting of the classical operators on fuzzy sets. We define

$$M_{C(\widetilde{A_1},\widetilde{A_2})}(x) = I_2 - M_{\widetilde{A_1},\widetilde{A_2}}(x) \tag{10.1}$$

and it is easy to verify that

$$M_{\widetilde{A_1},cl(\widetilde{A_1})}(x) = \frac{1}{2}\begin{pmatrix} 1 & 2\mu_{\widetilde{A_1}}(x)-1 \\ 2\mu_{\widetilde{A_1}}(x)-1 & 1\end{pmatrix} \tag{10.2}$$

Furthermore, using (9.6) and (8.20) we have:

Proposition 10.1. *Let $\widetilde{A_1},\widetilde{A_2},\widetilde{B_1}$ and $\widetilde{B_2}$ be fuzzy sets with membership functions $\widetilde{A_1},\widetilde{A_2},\widetilde{B_1}$ and $\widetilde{B_2}$ respectively. Then we have*

$$M_{\widetilde{A_1},\widetilde{A_2}}(x)M_{\widetilde{B_1},\widetilde{B_2}}(x) = \mu_{\widetilde{A_1}}(x)\mu_{\widetilde{B_1}}(x)P + \mu_{\widetilde{A_2}}(x)\mu_{\widetilde{B_2}}(x)Q. \tag{10.3}$$

Thus the matrix product of the hyperbolic membership functions $M_{\widetilde{A_1},\widetilde{A_2}}$ and $M_{\widetilde{B_1},\widetilde{B_2}}$ corresponds to the algebraic product (See Section 5 and [10, §3.3. p. 33]) of the fuzzy sets $\widetilde{A_1}$ and $\widetilde{B_1}$ along P and $\widetilde{A_2}$ and $\widetilde{B_2}$ along Q.

Proposition 10.2. *In the above notations it holds that:*

$$M_{\widetilde{A_1},\widetilde{A_2}}(x)M_{\widetilde{A_2},\widetilde{A_1}}(x) = \mu_{\widetilde{A_1}}(x)\mu_{\widetilde{A_2}}(x)I_2 \tag{10.4}$$

Proof.

$$M_{\widetilde{A_1},\widetilde{A_2}}(x)M_{\widetilde{A_2},\widetilde{A_1}}(x) = \mu_{\widetilde{A_1}}(x)\mu_{\widetilde{A_2}}(x)P + \mu_{\widetilde{A_2}}(x)\mu_{\widetilde{A_1}}(x)Q = \mu_{\widetilde{A_1}}(x)\mu_{\widetilde{A_2}}(x)I_2$$

□

By (8.9), we have

$$M_{\widetilde{A_1},\widetilde{A_2}}(x) \vee M_{\widetilde{B_1},\widetilde{B_2}}(x) = (\mu_{\widetilde{A_1}}(x) \vee \mu_{\widetilde{B_1}}(x))P + (\mu_{\widetilde{A_2}}(x) \vee \mu_{\widetilde{B_2}}(x))Q. \tag{10.5}$$

11. Betweenness for hyperbolic-valued membership functions

The proofs of the results in this section are easily adapted from the proofs in the scalar case by considering the idempotent decomposition, as in previous arguments in the paper, and we will not write out the details.

Definition 11.1. Let $M_{\widetilde{A_1},\widetilde{A_2}}$, $M_{\widetilde{B_1},\widetilde{B_2}}$ and $M_{\widetilde{C_1},\widetilde{C_2}}$ be three hyperbolic-valued membership functions defined on the set X. One says that $M_{\widetilde{C_1},\widetilde{C_2}}$ *is pointwise between* $M_{\widetilde{A_1},\widetilde{A_2}}$ *and* $M_{\widetilde{B_1},\widetilde{B_2}}$, if

$$M_{\widetilde{A_1},\widetilde{A_2}}(x)\wedge M_{\widetilde{B_1},\widetilde{B_2}}(x)\le M_{\widetilde{C_1},\widetilde{C_2}}(x)\le M_{\widetilde{A_1},\widetilde{A_2}}(x)\vee M_{\widetilde{B_1},\widetilde{B_2}}(x),\quad x\in X. \tag{11.1}$$

In the setting of $\mathbb{D}$-valued membership functions Lemma 4.2 and Proposition 6.2 become:

Proposition 11.2. *Let* $M_{\widetilde{A_1},\widetilde{A_2}}, M_{\widetilde{B_1},\widetilde{B_2}}$ *and* $M_{\widetilde{C_1},\widetilde{C_2}}$ *be* $\mathbb{D}$*-valued membership functions. Then* $M_{\widetilde{C_1},\widetilde{C_2}}$ *is pointwise between* $M_{\widetilde{A_1},\widetilde{A_2}}$ *and* $M_{\widetilde{B_1},\widetilde{B_2}}$ *if and only if*

$$M_{\widetilde{C_1},\widetilde{C_2}}(x)=M_{\widetilde{A_1},\widetilde{A_2}}(x)\wedge M_{\widetilde{B_1},\widetilde{B_2}}(x)+M_{\widetilde{Z_1},\widetilde{Z_2}}(x), \tag{11.2}$$

where $M_{\widetilde{C_1},\widetilde{C_2}}(x)$ *is a* $\mathbb{D}$*-valued membership function satisfying*

$$M_{\widetilde{Z_1},\widetilde{z_2}}(x)\le M_{\widetilde{A_1},\vee A_2}(x)\vee M_{\widetilde{B_1},\widetilde{B_2}}(x)-M_{\widetilde{A_1},\widetilde{A_2}}(x)\wedge M_{\widetilde{B_1},\widetilde{B_2}}(x),\quad x\in X. \tag{11.3}$$

Furthermore, the idempotent decomposition (8.17) gives:

Proposition 11.3. *In the notation of the previous proposition,* $M_{\widetilde{C_1},\widetilde{C_2}}$ *is pointwise between* $M_{\widetilde{A_1},\widetilde{A_2}}$ *and* $M_{\widetilde{B_1},\widetilde{B_2}}$ *if and only if* $\widetilde{C_1}$ *and* $\widetilde{C_2}$ *are pointwise between* $\widetilde{A_1}$ *and* $\widetilde{B_1}$ *and* $\widetilde{A_2}$ *and* $\widetilde{B_2}$ *respectively.*

The hyperbolic counterparts of Definition 7.1 and Theorem 7.2 in the hyperbolic setting are:

Definition 11.4. Let $\mathbf{a}\in\mathbb{D}$. The $\mathbf{a}$-cut associated to the $\mathbb{D}$-valued membership function $M_{\widetilde{A_1},\widetilde{A_2}}$ is the set of elements $M^{-1}_{\widetilde{A_1},\widetilde{A_2}}$.

Theorem 11.5. $M_{\widetilde{C_1},\widetilde{C_2}}$ *is* $\mathbf{a}$*-between* $M_{\widetilde{A_1},\widetilde{A_2}}$ *and* $M_{\widetilde{B_1},\widetilde{B_2}}$ *if and only if* $M_{\widetilde{C_1},\widetilde{C_2}}$ *is pointwise between* $M_{\widetilde{A_1},\widetilde{A_2}}$ *and* $M_{\widetilde{B_1},\widetilde{B_2}}$.

We conclude with a counterpart of Theorem 7.3 for hyperbolic-valued membership functions. The novelty is what one now needs the $\mathbb{H}$-valued counterpart of a distance to get a triangle equality. Here too the proof is easy, going via the idempotent decomposition (8.17), and will be omitted. With $D(\mu_{\widetilde{A}},\mu_{\widetilde{B}})$ as in (7.13) we define

$$D_{\mathbb{H}}(M_{\widetilde{A_1},\widetilde{A_2}},M_{\widetilde{B_1},\widetilde{B_2}})=U\begin{pmatrix}D(\mu_{\widetilde{A_1}},\mu_{\widetilde{B_1}}) & 0\\ 0 & D(\mu_{\widetilde{A_2}},\mu_{\widetilde{B_2}})\end{pmatrix}U. \tag{11.4}$$

Theorem 11.6. *Assuming the integral well defined,* $M_{\widetilde{C_1},\widetilde{C_2}}$ *is pointwise between* $M_{\widetilde{A_1},\widetilde{A_2}}$ *and* $M_{\widetilde{B_1},\widetilde{B_2}}$ *if and only if the equality holds*

$$D_{\mathbb{H}}(M_{\widetilde{A_1},\widetilde{A_2}},M_{\widetilde{B_1},\widetilde{B_2}})=D_{\mathbb{H}}(M_{\widetilde{A_1},\widetilde{A_2}},M_{\widetilde{C_1},\widetilde{C_2}})+D_{\mathbb{H}}(M_{\widetilde{C_1},\widetilde{C_2}},M_{\widetilde{B_1},\widetilde{B_2}}).$$

12. Conclusion

Hyperbolic numbers had recently numerous applications to machine learning (See e.g. [14, 15] and allow the extension of the notion of probability to a wider setting, where a probability has two components rather than being a scalar number (See [16]). Combined with the notion of fuzzy sets one gets, as illustrated in the present paper, new ways to consider uncertainty.

Statements and Declarations of Competing Interests. We declare that we have no conflict of interests.

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References

[1] P. Jaccard, Distribution de la flore alpine dans le bassin des dranses et dans quelques régions voisines, Bull. Soc. Vaudoise Sci. Nat. 37 (1901) 241–272.
[2] Y. Horibe, A note on entropy metrics, Information and Control 22 (1973) 403–404.
[3] C. Rajski, A metric space of discrete probability distributions, Information and Control 4 (1961) 371–377.
[4] E. Marczewski and H. Steinhaus, On a certain distance of sets and the corresponding distance of functions, In Colloquium Mathematicum, volume 6, pages 319–327. Instytut Matematyczny Polskiej Akademii Nauk, 1958.
[5] E. Marczewski and H. Steinhaus, On the systematik distance of biotopes, Zastosow. Mat. 4 (1959) 195–202.
[6] M. Levandowsky and D. Winter, Distance between sets, Nature, 234(5323):34–35, 1971.
[7] A. Tversky, Features of similarity, Psychological review 84 (4) (1977) 327–352.
[8] Shihyen Chen, Bin Ma and Kaizhong Zhang, On the similarity metric and the distance metric, Theoret. Comput. Sci. 410 (24-25) (2009) 2365–2376.
[9] F. Restle, A metric and an ordering on sets, Psychometrika 24 (3) (1959) 207–220.
[10] C. Mohan. An introduction to fuzzy set theory and fuzzy logic, MV Learning, second edition, 2019.
[11] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[12] I. Aizenberg, N. Aizenberg and J. Vandewalle, Multi-valued and universal binary neurons: Theory, learning and applications. Kluwer Academic Publishers, 2000.
[13] Y. Koroe, A model of complex-valued associative memories and its dynamics, In Akira Hirose, editor, Complex-valued neural networks, pages 57–79. World Scientific, 2003.
[14] D. Alpay, K. Diki, and M. Vajiac, A note on the complex and bicomplex valued neural networks, Appl. Math. Comput., 445:Paper No. 127864, 12, 2023.
[15] M. Kobayashi, Hyperbolic Hopfield neural networks, IEEE Transactions On Neural Networks and Learning Systems 24 (2) (2013) 335–341.
[16] D. Alpay, M. E. Luna-Elizarrarás and M. Shapiro, Kolmogorov's axioms for probabilities with values in hyperbolic numbers, Adv. Appl. Clifford Algebr. 27 (2) (2017) 913–929.
[17] T. A. Springer, Linear algebraic groups, volume 9 of Progress in Mathematics. Birkhäuser Boston, Mass., 1981.
[18] D. Alpay. The Schur algorithm, reproducing kernel spaces and system theory, American Mathematical Society, Providence, RI, 2001. Translated from the 1998 French original by Stephen S. Wilson, Panoramas et Synthèses.
[19] N. Aronszajn, Theory of reproducing kernels, Trans. Amer. Math. Soc. 68 (1950) 337–404.
[20] S. Saitoh, Theory of reproducing kernels and its applications, volume 189. Longman scientific and technical, 1988.
[21] S. Pereverzyev, An Introduction to Artificial Intelligence Based on Reproducing Kernel Hilbert Spaces. Springer Nature, 2022.

[22] D. Alpay and P. Jorgensen, New characterizations of reproducing kernel Hilbert spaces and applications to metric geometry, Opuscula Math. 41 (3) (2021) 283–300.

[23] W. L. Hays, An approach to the study of trait implication and trait similarity, In R. Tagiuri and L. Petrullo, editors, Person perception and interpersonal behavior, pages 289–299. Stanford University Press Stanford, 1958.

[24] R. Zwick, E. Carlstein and D. V. Budescu, Measures of similarity among fuzzy concepts: A comparative analysis, International Journal of Approximate Reasoning, 1 (2) (1987) 221–242.

[25] C. E. Shannon, A mathematical theory of communication, Bell System Tech. J. 27 (1948) 379–423, 623–656.

[26] J. Łukasiewicz, Selected works, Edited by L. Borkowski, Studies in Logic and the Foundations of Mathematics. Amsterdam etc.: North-Holland Publishing Company; Warszawa: PWN - Polish Scientific Publishers. xii, 405 pages, 1970.

[27] G. Chen and T. Pham, Introduction to fuzzy sets, fuzzy logic and fuzzy control systems, CRC Press LLC, 2001.

[28] H. X. Li and V. C. Yen, Fuzzy sets and fuzzy decision-making. CRC Press, Boca Raton, FL, 1995.

[29] W. Pedrycz and F. Gomide, An introduction to fuzzy sets: analysis and design, With a foreword by Lotfi A. Zadeh. Cambridge, MA: MIT Press, 1998.

[30] H.-J. Zimmermann, Fuzzy set theory and its applications. Kluwer Academic Publishers, Boston, MA, second edition, 1992. With a foreword by L. A. Zadeh.

[31] J. A. Goguen. L-fuzzy sets, J. Math. Anal. Appl. 18 (1967) 145–174.

[32] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 (1) (1986):87–96.

[33] A. L. Ralescu and D. A. Ralescu, Probability and fuzziness, Inform.n Sci. 34 (2) (1984) 85–92.

[34] D. Alpay, M. Luna-Elizarrarás, M. Shapiro and D. C. Struppa, Basics of functional analysis with bicomplex scalars, and bicomplex Schur analysis, Springer Briefs in Mathematics. Springer, Cham, 2014.

[35] M. E. Luna-Elizarrarás, M. Shapiro, D. C. Struppa and A. Vajiac, Bicomplex holomorphic functions, Frontiers in Mathematics. Birkhäuser/Springer, Cham, 2015. The algebra, geometry and analysis of bicomplex numbers.

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